Published by Institute of Physics Publishing for SISSA/ISAS

Received: *December 21, 2001*
Revised: *April 25, 2002*
Accepted: *May 8, 2002*

Hopf algebra of ribbon graphs and renormalization

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Abstract: Connes and Kreimer have discovered a Hopf algebra structure behind renormalization of Feynman integrals. We generalize the Hopf algebra to the case of ribbon graphs, i.e. to the case of theories with matrix fields. The Hopf algebra is naturally defined in terms of surfaces corresponding to ribbon graphs. As an example, we discuss renormalization of Φ^4 theory and the $1/N$ expansion.

Keywords: Renormalization Regularization and Renormalons, 1/N Expansion.

Contents

1. Introduction

Connes and Kreimer have discovered a Hopf algebra of graphs [1]. This algebra underlies Bogolubov's recursive formula [4] for the renormalization of Feynman integrals. The renormalization becomes more clear and natural from the algebraic point of view. The Hopf algebra of graphs is dual to the universal enveloping algebra of some Lie algebra. This Lie algebra generates diffeomorphisms in the space of coupling constants [2, 5, 6].

In this paper we construct the Hopf algebra of ribbon graphs. Ribbon graphs appear in theories with matrix fields. We cannot permute any two matrices in a product, thus we are not allowed to permute any two edges of the vertex of interaction. If the lagrangian has the form of a trace of matrices, then there are only cyclic permutations of lines at any vertex of the ribbon graph. In the paper we consider a more general situation. We assume that the lines attached to any vertex of the ribbon graph can be divided into groups. In any group cyclic permutations of the lines are allowed. We also admit any permutations of the groups. In this case the interaction has the form of a product of traces.

It is known that any ribbon graph corresponds to surfaces with a boundary. The ribbon graph can be drawn without self-intersections on any corresponding surface. There are infinitely many such surfaces but the surface with the maximal Euler characteristic is unique. We shall describe a one-to-one correspondence between ribbon graphs and surfaces endowed with additional structures. These structures have the form of cell decompositions.

In any cell decomposition the two dimensional cells are open disks. In our construction we use open spheres with holes instead of open disks. This structure will be called a quasi-cell decomposition. After that we construct a Hopf algebra of the surfaces with quasi-cell decompositions. This algebra immediately gives us Bogolubov's recursive formula in theories with matrix fields. The Hopf algebra of surfaces and the corresponding Lie algebra may show up in string theories.

The plan of the paper is as follows. In section 2 we define ribbon graphs and corresponding surfaces with quasi-cell decompositions. As an example, we consider the ϕ^4 theory and discuss the $1/N$ expansion.

In section 3 we deal with the corresponding surfaces. At first we introduce necessary definitions and essential operations. After that we define the Hopf algebra.

In section 4 we consider dual objects to the Hopf algebra. The dual space is the space of linear functions on the Hopf algebra. This space is isomorphic to the universal enveloping algebra of a Lie algebra. The generators of the Lie algebra correspond to 1PI graphs [1]. After that we introduce the group of characters of the Hopf algebra. Any Quantum Field Theory can be viewed as a character of the Hopf algebra of graphs. Using the algebraic methods due to Connes and Kreimer [1], we obtain the renormalization of Feynman integrals. The last section is devoted to the appendixes, we give proofs and special constructions in this section.

2. Ribbon graphs

Any ribbon graph consists of vertices connected by lines. Two vertices can be connected by several lines, and a line can begin and finish at the same vertex. In the latter case we say that the line makes a loop at the vertex. Also a graph can be a disjoint union of connected graphs. The vertices of any ribbon graph are called ribbon vertices. In general, only cyclic permutations of the lines are allowed in the ribbon vertices.

Let Γ be a ribbon graph and let $C \in \Gamma$ be a ribbon vertex. We denote by $\Lambda_C = \{l_1, l_2, \ldots, l_n\}$ the set of all lines attached to the vertex. If a line l_i makes a loop at C, then the set Λ_C contains this line twice: $\Lambda_C = \{l_1, \ldots, l_i', \ldots, l_i'', \ldots, l_n\}$. The order of elements of Λ_C is the order in which the lines are attached to the vertex. The number of elements of Λ_C is called the degree of $C \in \Gamma$ and is denoted by $\deg_\Gamma(C)$. If p is the number of lines with one of the ends at C and q is the number of lines with the both ends at C, then

$$\deg_\Gamma(C) = p + 2q\,.$$

A ribbon vertex C will be called simple if the order of elements of Λ_C is fixed and only cyclic permutations are allowed. For instance, $\{l_1, l_2, \ldots, l_n\} \sim \{l_n, l_1, \ldots, l_{n-1}\}$. The corresponding interaction has the form of a trace.

A ribbon vertex C will be called complex if the set Λ_C can be divided into subsets Λ_C^i, $i = 1, \ldots, m$ such that $\Lambda_C = \{\Lambda_C^1, \Lambda_C^2, \ldots, \Lambda_C^m\}$. In each subset the order of elements is fixed and cyclic permutations are allowed. In Λ_C any permutations of the subsets are allowed: $\{\dots,\Lambda^i_C,\dots,\Lambda^j_C,\dots\}\sim\{\dots,\Lambda^j_C,\dots,\Lambda^i_C,\dots\}$. We denote by $\deg^i_\Gamma(C)$ the number of elements of the subset Λ^i_C. We see that

$$\deg_\Gamma(C)=\sum_{i=1}^{m}\deg^i_\Gamma(C)\,.$$

The corresponding interaction has the form of a product of traces.

Example. Consider the theory of scalar complex fields $\Phi(x)$ in 4-dimensional euclidean space with the lagrangian

$$\begin{aligned} L &= L_0+L_{int}\,,\\ L_0 &= Tr\left(\partial\Phi^+\partial\Phi+m^2\Phi^+\Phi\right),\\ L_{int} &= g_1\,Tr\left[\left(\Phi^+\Phi\right)^2\right]+g_2\,\left(Tr\left[\Phi^+\Phi\right]\right)^2. \end{aligned}\tag{2.1}$$

Here we have denoted

$$\partial\Phi^+\partial\Phi=\sum_{\mu=1}^{4}\partial_\mu\Phi^+(x)\partial_\mu\Phi(x)\,.$$

The field $\Phi(x)$ is a complex $N\times N$ matrix. We have

$$L_0 = \partial\Phi^{+i}_{\;j}\,\partial\Phi^j_i+m^2\Phi^{+i}_{\;j}\,\Phi^j_i\,,\tag{2.2}$$

$$L_{int} = g_1\,\Phi^{+i}_{\;j}\,\Phi^j_k\,\Phi^{+k}_{\;l}\,\Phi^l_i+g_2\,\left(\Phi^{+i}_{\;j}\,\Phi^j_i\right)\left(\Phi^{+k}_{\;l}\,\Phi^l_k\right).\tag{2.3}$$

There is a summation over all the repeating indices $i,j,k,l=1,\dots,N$.

The propagators in the theory $<\Phi^{+i}_{\;j}\,\Phi^k_l>\sim\delta^i_l\,\delta^k_j$. For brevity we do not discuss the momentum dependence here. In Feynman diagrams the propagators have the form as in figure 1.

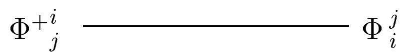

Figure 1: Propagator.

It is convenient to represent the propagators as consisting of two threads with arrows. We also put indices and signs at the ends (see figure 2).

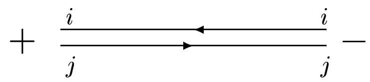

Figure 2: Propagator.

The sign $'+'$ corresponds to the field Φ^+, the sign $'-'$ corresponds to the field Φ. The incoming arrow corresponds to the upper index, the outgoing arrow corresponds to the lower index.

The interaction vertices in the theory are as in figure 3.

The degree of the vertices is equal to four. The left vertex is simple. The right one is complex, indeed, the set of lines attached to the vertex consists of two subsets with two lines in each subset.

We see that the structure of vertices is rather complicated in ribbon graphs. We can consider two-dimensional objects instead of the vertices. This enables us to describe the different structures of vertices more efficiently.

Now we construct surfaces corresponding to ribbon graphs. Let Γ be a ribbon graph. For simplicity, assume that all the vertices in Γ are simple. Let C be a vertex of Γ, we

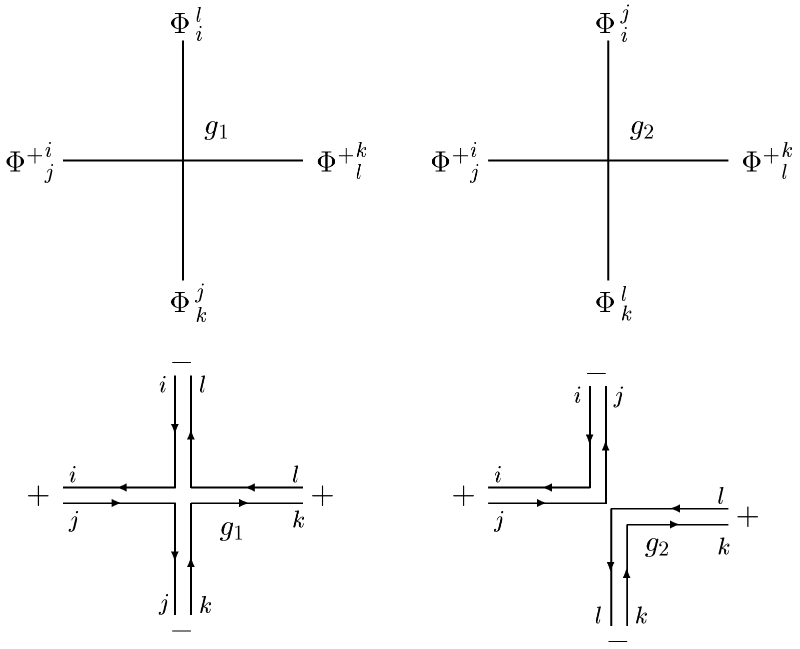

Figure 3: Interaction vertices.

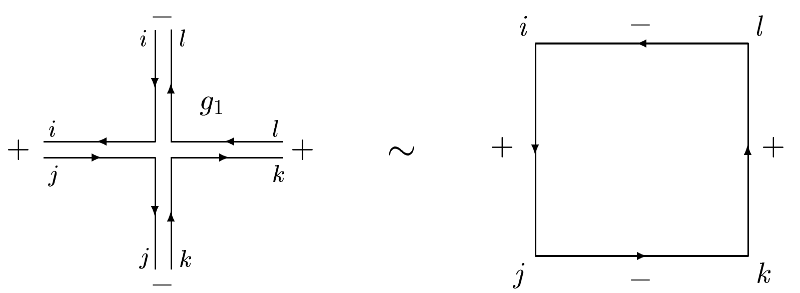

Figure 4: Vertex-polygon correspondence.

associate to this vertex a polygon with $\deg_\Gamma(C)$ edges. For example, any simple vertex with four attached lines corresponds to a square.

Any line at the vertex corresponds to an edge of the polygon. We put indices, signs and arrows of the line on the corresponding edge.

A line between two vertices is a common edge between the corresponding polygons. We put indices near the ends of edges of the polygons (see figure 4). We have only one index near any such point. This property remains after the connection of the polygons. For instance, we have the index j near the vertex A and the index m near the vertex B (see figure 5).

In ribbon graphs the indices are associated with the threads. Consequently the points at the ends of edges correspond to the threads in ribbon graphs. We shall denote by indices the corresponding threads or points at the ends of edges. For instance, we shall say 'the thread k' or 'the point j'.

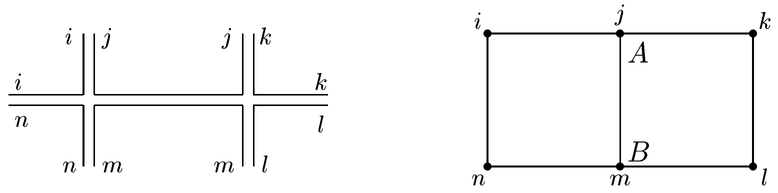

Figure 5: Connection of vertices-polygons.

If a line makes a loop at a vertex, then we connect the edges of the corresponding polygon (see figure 6).

Thus we can find the corresponding surface with a boundary for any ribbon graph with simple vertices. The internal lines of the graph correspond to the internal edges on the surface, the external lines correspond to the external edges (the edges on the boundary of the surface). Note that the surface is a CW-complex in this case.

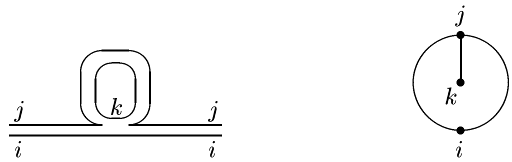

Figure 6: Index loop.

Let us recall the definition of the CW-complex. Any CW-complex is a union of cells. We denote by $C^{(n)}$ an n-dimensional cell. Zero-dimensional cells are points, one-dimensional cells are open segments, two-dimensional cells are diffeomorphic to open disks, n-dimensional cells are diffeomorphic to open n-dimensional balls for $n > 0$. Any N-dimensional CW-complex $\mathcal{M}_N$ is a union

$$\mathcal{M}_N = \bigcup_{n=0}^{N} \bigcup_{i_n} C_{i_n}^{(n)} \tag{2.4}$$

such that the boundary of any k-dimensional cell $C_{i_k}^{(k)}$ for all $k \in \{1, \ldots, N\}$ belongs to

$$\mathcal{M}_{k-1} = \bigcup_{n=0}^{k-1} \bigcup_{i_n} C_{i_n}^{(n)} \subset \mathcal{M}_N \,,$$

$$fr\, C_{i_k}^{(k)} \subset \mathcal{M}_{k-1} \,.$$

For simplicity, we assume that union (2.4) is finite.

For example, suppose we have a square, the boundary of any edge consists of the two points at the ends, the interior of the square has the boundary consisting of the four edges. Thus any square is a two-dimensional CW-complex.

We see that if we construct a surface by identifying, or gluing, edges of polygons, then this surface is a CW-complex. In this case we also say that the polygons give the cell decomposition of the surface.

Remark. Let us consider figure 6. There is a closed thread inside the loop. Hence we have a sum over index k in the Feynman integral associated with this diagram. The sum gives the factor of N. Closed threads are also called index loops. The fact that the index loops give the factors of N is exploited in the $1/N$ expansion ('t Hooft [3]). We see that the number of index loops is an important characteristic of ribbon graphs.

We are going to find the objects on our surfaces which correspond to the index loops. Let S be a surface with a cell decomposition. Let j be a common end of the edges $l_1^j, \ldots, l_n^j \subset S$. Since the surface is closed, the point j lies on the boundary if and only if at least one of the edges $l_1^j, \ldots, l_n^j$ is external. Indeed, if the vertex is external, then by continuity there exists an external edge attached to this vertex. On the other hand, if an edge is external, then the both ends of this edge are external vertices.

Since the open threads end at external lines, they correspond to points on the boundary of the corresponding surface. The closed threads go along internal lines, hence they correspond to internal points. Thus index loops correspond to the internal points at the ends of edges.

Let Γ be a ribbon graph without external lines and let S_Γ be the corresponding surface. This surface has no boundary. We can draw the graph on the surface without self intersections. Indeed, we can put each vertex of the graph at the center of the corresponding polygon. The lines of the graph intersect the corresponding edges on the surface.

We have noted that the polygons give a cell decomposition of the surface. We see that the graph gives another cell decomposition of the surface. By construction, the number of vertices of the graph equals the number of polygons, i.e. two-dimensional cells. The lines of the graph and the edges of the polygons are one-dimensional cells, we have the same numbers of lines and edges. One can view the threads of the ribbon graph as the 'boundaries' of faces, then the number of faces is equal to the number of index loops. The index loops correspond to the points at the ends of edges. We see that the number of zero-dimensional cells of one decomposition is equal to the number of two-dimensional cells of the other; the numbers of one-dimensional cells are the same. Such two cell decompositions are called dual.

We denote by V the number of vertices of the graph Γ, by P the number of lines, and by I the number of index loops. Note that in our example the graph Γ has no external lines and the surface S_Γ has no boundary. We have V polygons, P edges and I points at the ends of edges on S_Γ. Let χ be the Euler characteristic of S_Γ. The Euler formula in this case is

$$V - P + I = \chi\,.$$

The last equation is very important in the $1/N$ expansion. But if there are complex ribbon vertices, then this formula is not so simple.

Let C be a complex vertex of a ribbon graph Γ. Let Λ_C be the set of lines attached to C. Suppose that Λ_C is divided into m subsets Λ_C^i, $i = 1, \ldots, m$, then we associate to the vertex a sphere with m holes. 'A sphere with holes' means any manifold, topologically equivalent to a sphere with holes. Note that a sphere with one hole is equivalent to a polygon. The subset Λ_C^j corresponds to the j-th hole. The boundary of this hole consists of $\deg_\Gamma^j(C)$ edges, i.e. the number of edges in the j-th hole is equal to the number of lines of the corresponding j-th subset. A line between two vertices is a common edge between the two corresponding spheres with holes. We see that the surface corresponding to the graph Γ has a decomposition into spheres with holes. This decomposition resembles the cell decomposition but open spheres with holes are not diffeomorphic to open disks. Thus the decomposition into spheres with holes will be called a quasi-cell decomposition. The corresponding surface will be called a surface with quasi-cell decomposition or simply a surface with decomposition.

Let S_Γ be the surface with decomposition corresponding to a ribbon graph Γ. We denote by V, P, I the numbers of vertices, lines, and index loops of Γ respectively. Then in S_Γ we have V spheres with holes, P internal edges and I internal points at the ends of edges. Let m_k be the number of holes in the k-th sphere, $k = 1, \ldots, V$. We assume that $m_k \geq 1$. We can put $(m_k - 1)$ edges on the k-th sphere in order to obtain a cell decomposition of this sphere. Such decomposition of all spheres yields a cell decomposition of S_Γ. If χ is the Euler characteristic of S_Γ, then the Euler formula is

$$I - P - \sum_{k=1}^{V}(m_k - 1) + V = \chi\,,$$

or

$$I - P + \sum_{k=1}^{V}(2 - m_k) = \chi\,.$$

The numbers of points and edges on the boundary cancel each other and we do not write them in the formulas.

Denote by χ_{m_k} the Euler characteristic of the sphere with m_k holes. Since $\chi_{m_k} = 2 - m_k$, we have

$$I - P + \sum_{k=1}^{V} \chi_{m_k} = \chi\,. \tag{2.5}$$

The last formula is nontrivial from the point of view of graphs.

Example. Let us consider the $1/N$ expansion in the theory with lagrangian (2.1).

Let Γ be a ribbon graph in this theory. Suppose that we have V_1 vertices g_1 and V_2 vertices g_2. The corresponding Euler characteristics are $\chi_1 = 1$ and $\chi_2 = 0$. Let P be the number of internal lines, R be the number of external lines, I be the number of index loops, and χ be the Euler characteristic of the corresponding surface. We associate to Γ the number

$$X_\Gamma = N^I \cdot g_1^{V_1} \cdot g_2^{V_2}\,.$$

The numbers V_1, V_2, P, R are not independent. Indeed, we can put points near the internal ends of the lines. Then there are four points near every vertex, any internal line has two points, any external line has one point. The number of points is

$$4(V_1 + V_2) = 2P + R\,. \tag{2.6}$$

We have

$$P = 2(V_1 + V_2) - R/2\,. \tag{2.7}$$

Using (2.5)

$$I - P + \sum_{k=1}^{V_1} \chi_1 + \sum_{l=1}^{V_2} \chi_2 = \chi\,,$$

or

$$I - P + V_1 = \chi\,.$$

We replace P by $2(V_1 + V_2) - R/2$ in the last formula and obtain

$$I - V_1 - 2V_2 = \chi - R/2\,.$$

If we assume that $g_1 = g_1^0/N$ and $g_2 = g_2^0/N^2$, where g_1^0 and g_2^0 do not depend on N, then

$$X_\Gamma = N^{I-V_1-2V_2} \cdot (g_1^0)^{V_1} \cdot (g_2^0)^{V_2} \sim N^{\,\chi - R/2}. \tag{2.8}$$

If the number of external lines is fixed, then X_Γ depends only on topological characteristics of the corresponding surface.

We see that for a given ribbon graph we can construct the corresponding surface. Now, assume that a surface consists of spheres with holes connected along the edges. We can always find the ribbon graph corresponding to this surface. Hence we have the one-to-one correspondence between the ribbon graphs and the surfaces with decomposition into spheres with holes.

3. Surfaces

In this section we consider surfaces with a boundary. The surfaces have decompositions into spheres with holes. As above, 'sphere with holes' means any manifold topologically equivalent to a sphere with holes. We assume, that any sphere with holes has edges on its boundary.

Let C be a sphere with holes. The number of edges of C is called the degree of C and is denoted by $\deg(C)$. Let m be the number of holes of C. We denote by $\deg^i(C)$ the number of edges on the boundary of the i-th hole, $i = 1, \dots, m$. We see that $\deg(C) = \sum_{i=1}^m \deg^i(C)$.

Let l_1 and l_2 be different edges; they may belong either to one sphere with holes or to different ones. We can identify l_1 and l_2; the edge $l \sim l_1 \sim l_2$ is internal. We do not identify internal edges with any other edges, i.e. one may identify only two (but not more) different external edges. If we take several spheres with holes and identify some pairs of edges, we obtain a surface. We allow the surface to be a disjoint union of connected components. We say that this surface has a decomposition into spheres with holes. Such surfaces will be also called surfaces with decomposition. In the previous section we have shown that such surfaces correspond to ribbon graphs.

Let S be a surface with decomposition, and let $C \subset S$ be a sphere with holes. We denote by $\deg_S(C)$ the degree of C inside S and define it by the formula $\deg_S(C) = \deg(C)$. Similarly, if C has several holes, then $\deg_S^i(C) = \deg^i(C)$.

Now we define the operation of extraction of a subsurface. Let S be a surface with decomposition, and let s be an open subsurface of S (the points on the boundary of s do not belong to s). Suppose that s is a disjoint union of connected components s^j, $j = 1, \ldots, J$.

There is a natural mapping of connected open surfaces to open spheres with holes. We define the mapping by the following rule. A surface with h holes maps to a sphere with h holes. Now let c^j be the open sphere with holes corresponding to s^j. We can extract s^j from S and insert c^j in place of s^j in the way that the boundary of s^j becomes the boundary of c^j. We denote by S/s the surface which we obtain after the extraction of all s^j and the insertion of the corresponding c^j.

Now we define the closure of s with respect to S. The closure of s is denoted by $\bar{s}$. We add edges on the boundary of s, so that every sphere with holes inside $\bar{s}$ has the same degree as it has had in S. In other words, let $C \subset S$ be a sphere with holes and let $\mathrm{int}(C) \subset s$.[1] We define $\bar{s}$ to be the closure of s such that $\deg_{\bar{s}}(C) = \deg(C) = \deg_S(C)$. If C has m holes, then $\deg^i_{\bar{s}}(C) = \deg^i(C)$, $i = 1, \ldots, m$.

If s is the empty surface $\emptyset$, then we define $S/\emptyset = S$. If $\bar{s} = S$, then we define $S/s = \emptyset$.

Also we use the following notation for the operation of extraction $\varphi_s : \; S \to S/s$.

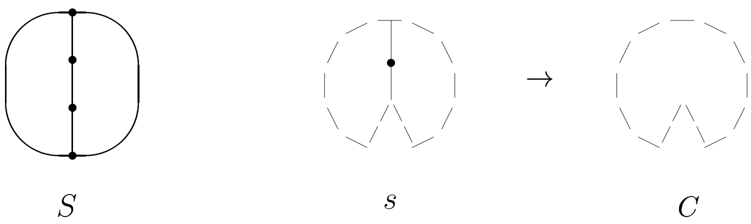

Figure 7: Extraction.

Example. In figure 7 s is an open subsurface of S and C is the corresponding 'open sphere with holes'. We see that C is a sphere with one hole.

In figure 8 S/s is the surface S with extracted s, the surface $\bar{s}$ is the closure of s with respect to S. Observe, that the number of internal edges on the surface S is the same as on S/s and $\bar{s}$ together. Indeed, we have three internal edges on S, one internal edge on S/s and two internal edges on $\bar{s}$. Also we have the conservation of the number of internal vertices, there are two of them on S, one on S/s, and one on $\bar{s}$. The third number, which is conserved, is the number of spheres with holes minus the number of connected components of the surface S.

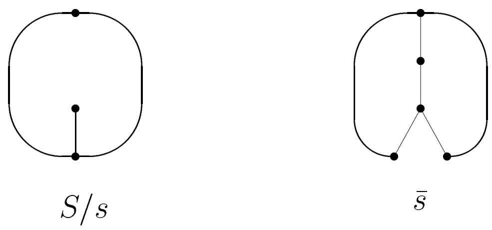

Figure 8: Result of extraction.

The corresponding ribbon graphs are planar and we can draw them as the ordinary ones (see figure 9).

[1]We denote by $\mathrm{int}(S)$ the interior of the surface S.

In graphs we have the conservation of the number of internal lines, of the number of index loops, and of the number of vertices minus the number of connected components of the graphs.

We denote by P_S the number of internal edges on the surface S, by $\mathrm{T}_S = \mathrm{C}_S - con(S)$ the number of spheres with holes minus the number of connected components of the surface S, and by I_S the number of internal points at the ends of internal edges on the surface S.

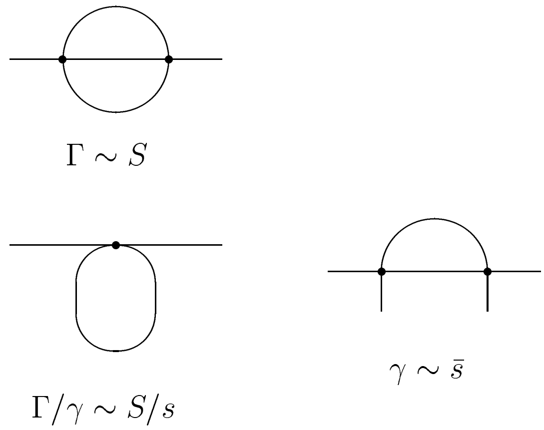

Figure 9: Extraction of graphs.

Remark. The operation of extraction conserves the numbers P_S, T_S and I_S

$$\begin{aligned}
\mathrm{P}_S &= \mathrm{P}_{\bar{s}} + \mathrm{P}_{S/s}\,, \\
\mathrm{T}_S &= \mathrm{T}_{\bar{s}} + \mathrm{T}_{S/s}\,, \\
\mathrm{I}_S &= \mathrm{I}_{\bar{s}} + \mathrm{I}_{S/s}\,.
\end{aligned}$$

We use the conservation of P_S in proofs by induction, while the conservation of I_S is important for the $1/N$ expansion.

In the construction of the Hopf algebra Connes and Kreimer use 1PI (one particle irreducible) graphs [1]. Similarly, we introduce 1PI surfaces. A surface with decomposition is called 1PI if it is connected, contains at least 1 internal edge and remains connected after the removal of any internal edge. Also we need the empty surface and the disjoint unions of 1PI surfaces. We call these surfaces allowed. Thus the set of allowed surfaces, denoted by $\{S_1, S_2, \ldots\}$, contains the empty surface, the 1PI surfaces and the disjoint unions of 1PI surfaces. Note that this set is countable.

Let S be an allowed surface. We can restrict the operation of extraction to allowed open subsurfaces $s_0 \subset S$. An open surface s is called allowed if the closure $\bar{s}$ is an allowed surface.

The operation of extraction has the following properties

1. Bijection. Let $s_0 \subset S$ be an open allowed subsurface. Consider the set of closed allowed surfaces $\{S_1 : s_0 \subset S_1 \subset S\}$. Since $s_0 \subset S_1$, we can define the surface $S_1/s_0 \subset S/s_0$. Hence we have the mapping $\varphi_{s_0} : \{S_1 : s_0 \subset S_1 \subset S\} \to \{S_1/s_0\}$. Also it is not difficult to prove that the set of the surfaces S_1/s_0 is equal to the set of allowed subsurfaces of S/s_0. Thus we have the bijective mapping
$$\varphi_{s_0} : \{S_1 : s_0 \subset S_1 \subset S\} \to \{S_2 : S_2 \subset S/s_0\}\,.$$

2. Composition law. Let s_1 and s_{12} be open subsurfaces of S such that $s_1 \subset s_{12}$. We define s_{12}/s_1 as the interior of $\bar{s}_{12}/s_1$. We denote by s_2 the surface s_{12}/s_1. Note that

$s_2 \subset S/s_1$. The composition law is

$$(S/s_1)/s_2 = S/s_{12}. \tag{3.1}$$

In terms of mappings we have

$$\varphi_{s_2} \circ \varphi_{s_1} = \varphi_{s_{12}}.$$

The mappings are bijective, thus for any given $s_1 \subset S$ and $s_2 \subset S/s_1$ there exists a unique $s_{12} \subset S$ that satisfies (3.1).

These properties are necessary for the construction of a consistent Hopf algebra.

Any Hopf algebra is a linear space endowed with the operations of a product, a unit, a coproduct, a counit, and an antipode. The Hopf algebra of surfaces is similar to the Hopf algebra of graphs defined by Connes and Kreimer in the paper [1].

The basis in the linear space is labeled by all allowed surfaces. It is convenient to designate the basis vectors by the same letters $S_1,\ S_2,\dots$ as the corresponding surfaces. We denote by $\mathcal{H}$ the linear space with the basis $\{S_1,\ S_2,\dots\}$ over the field $\mathcal{F}$ containing the field of rational numbers. By definition, any vector $V \in \mathcal{H}$ is a linear combination

$$V = \sum_{k=1}^{\infty} \lambda_k S_k\,,$$

where $\lambda_k \in \mathcal{F}$ and $\lambda_k \neq 0$ for a finite number of indices $k \in \mathbb{N}$. We use the notation $\mathbb{N} = \{1, 2, \dots\}$.

The product is a bilinear mapping $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$. The basis in $\mathcal{H} \otimes \mathcal{H}$ is given by $S_i \otimes S_j$, $i, j \in \mathbb{N}$. We define the product on basis vectors as a disjoint union of the corresponding surfaces:

$$m(S_i \otimes S_j) = S_i \coprod S_j\,. \tag{3.2}$$

We also use the notation $m(S_i \otimes S_j) = S_i \cdot S_j$. Obviously, the product m is associative and commutative.

Further we denote by Γ an arbitrary basis vector. When we say that Γ is a surface, we mean the surface corresponding to the basis vector. The letter Γ is chosen to keep in mind that the surfaces correspond to ribbon graphs.

Since

$$\Gamma \cdot \emptyset = \emptyset \cdot \Gamma = \Gamma$$

for any Γ, the empty surface $\emptyset$ is the unit element with respect to the product m.

The unit is a linear mapping $i : \mathcal{F} \to \mathcal{H}$. We define it by the formula

$$i(1) = \emptyset\,. \tag{3.3}$$

Consequently $i(\lambda) = \lambda\emptyset$ for all $\lambda \in \mathcal{F}$.

The linear space $\mathcal{H}$ together with the product m and the unit i is an algebra.

A linear space together with a coproduct and a counit is called a coalgebra.

The coproduct is a linear mapping $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$. We define the coproduct by the formula

$$\Delta\Gamma = \sum_{\gamma\subset\Gamma} \bar{\gamma} \otimes \frac{\Gamma}{\gamma}\,, \tag{3.4}$$

the sum is over all allowed open subsurfaces $\gamma \subset \Gamma$.

Example. If $\Gamma = \emptyset$, then the only subset $\gamma \subset \Gamma$ is $\emptyset$ and the coproduct is $\Delta\emptyset = \emptyset \otimes \emptyset$.

For $\Gamma \neq \emptyset$ we shall also use two other formulas. The first is

$$\Delta\Gamma = \Gamma \otimes \emptyset + \emptyset \otimes \Gamma + \sum_{\gamma\subset\Gamma}{}' \bar{\gamma} \otimes \frac{\Gamma}{\gamma}\,. \tag{3.5}$$

The prime near the sum means that the summation is over allowed open subsurfaces $\gamma \subset \Gamma : \bar{\gamma} \neq \Gamma$, $\gamma \neq \emptyset$. The second is

$$\Delta\Gamma = \Gamma \otimes \emptyset + \sum_{\gamma\subset\Gamma}^{\neq} \bar{\gamma} \otimes \frac{\Gamma}{\gamma}\,, \tag{3.6}$$

where the sum is over all open $\gamma \subset \Gamma$ such that $\bar{\gamma} \neq \Gamma$, note that $\gamma = \emptyset$ is allowed.

The coproduct is not cocommutative but coassociative

$$(\Delta \otimes id)\Delta\Gamma = (id \otimes \Delta)\Delta\Gamma\,, \qquad \forall\Gamma. \tag{3.7}$$

The proof of the coassociativity is given in appendix A.

The counit is a linear mapping $\varepsilon : \mathcal{H} \to \mathcal{F}$. We define the counit by the formula

$$\varepsilon(\Gamma) = \delta_{\Gamma,\emptyset} = \begin{cases} 1 & \text{if } \Gamma = \emptyset\,; \\ 0 & \text{if } \Gamma \neq \emptyset\,. \end{cases} \tag{3.8}$$

Observe, that the coproduct and the counit map the spaces in the opposite directions with respect to the product and the unit.

A linear space together with the structures of an algebra and of a coalgebra is called a bialgebra if the following conditions are satisfied

$$\begin{aligned} \Delta(\Gamma_1 \cdot \Gamma_2) &= \Delta\Gamma_1 \cdot \Delta\Gamma_2\,, \\ \varepsilon(\Gamma_1 \cdot \Gamma_2) &= \varepsilon(\Gamma_1)\varepsilon(\Gamma_2). \end{aligned}$$

It is not difficult to prove using (3.4) and (3.8) that these conditions hold in the algebra of graphs.

The antipode is a homomorphism $S : \mathcal{H} \to \mathcal{H}$. We define the antipode by the formula

$$m(S \otimes id)\Delta(\Gamma) = i\varepsilon(\Gamma)\,. \tag{3.9}$$

We denote by id the identity mapping. The tensor product of mappings acts in the following way $(S \otimes id)(\Gamma_1 \otimes \Gamma_2) = S(\Gamma_1) \otimes \Gamma_2$.

In the case $\Gamma = \emptyset$ we have

$$\begin{aligned} i\varepsilon(\emptyset) &= i(1) = \emptyset\,; \\ m(S \otimes id)\Delta(\emptyset) &= m((S \otimes id)(\emptyset \otimes \emptyset) = S(\emptyset) \cdot \emptyset = S(\emptyset)\,. \end{aligned}$$

Hence $S(\emptyset) = \emptyset$.

Let $\Gamma \neq \emptyset$. Then

$$\begin{aligned} i\varepsilon(\Gamma) &= i(0) = 0\,; \\ m(S \otimes id)\Delta(\Gamma) &= m\left((S \otimes id)\left(\Gamma \otimes \emptyset + \emptyset \otimes \Gamma + {\sum_{\gamma \subset \Gamma}}' \bar{\gamma} \otimes \frac{\Gamma}{\gamma}\right)\right) \\ &= S(\Gamma) \cdot \emptyset + S(\emptyset) \cdot \Gamma + {\sum_{\gamma \subset \Gamma}}' S(\bar{\gamma}) \cdot \frac{\Gamma}{\gamma} \\ &= S(\Gamma) + \Gamma + {\sum_{\gamma \subset \Gamma}}' S(\bar{\gamma}) \cdot \frac{\Gamma}{\gamma}\,. \end{aligned}$$

Consequently we have the recursive formula

$$S(\Gamma) = -\Gamma - {\sum_{\gamma \subset \Gamma}}' S(\bar{\gamma}) \cdot \frac{\Gamma}{\gamma}\,, \tag{3.10}$$

or using (3.6)

$$S(\Gamma) = -\sum_{\gamma \subset \Gamma}^{\neq} S(\bar{\gamma}) \cdot \frac{\Gamma}{\gamma}\,, \tag{3.11}$$

The recursion is on the number of internal edges. Indeed, since $\bar{\gamma} \neq \Gamma$ on the right hand side, we have

$$P_{\bar{\gamma}} = P_\Gamma - P_{\Gamma/\gamma} < P_\Gamma\,.$$

The base of the recursion is $S(\emptyset) = \emptyset$, because $P_\Gamma \geq 0$ and $P_\Gamma = 0 \Leftrightarrow \Gamma = \emptyset$ for any allowed surface Γ.

Proposition 1. *The antipode, defined recursively by (3.10), is a homomorphism of $\mathcal{H}$.*

We have to prove that

$$S(\gamma_1 \cdot \gamma_2) = S(\gamma_1) S(\gamma_2)\,.$$

for any allowed γ_1 and γ_2. The proof is given in appendix B.

Also the antipode S satisfies

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = i \circ \varepsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta$$

or

$$m(S \otimes \mathrm{id})(\Delta\Gamma) = i\varepsilon(\Gamma) = m(\mathrm{id} \otimes S)(\Delta\Gamma)\,.$$

This equation is called the axiom of the antipode. The axiom is proved in appendix C.

There are two nontrivial facts in the construction of the Hopf algebra $\mathcal{H}$. These facts are the coassociativity and the axiom of the antipode. The proof of these axioms is based on the two properties of the operation of extraction. These properties are the only required conditions. Thus we can choose another set of allowed subsurfaces, and if the properties are satisfied, then we can construct another Hopf algebra.

Example. Consider the theory with lagrangian (2.1). We construct the Hopf algebra of the surfaces corresponding to the ribbon graphs in this theory. We have two types of vertices in the theory. The vertices g_1 correspond to squares, the vertices g_2 correspond to cylinders with two edges on each boundary. Also it is convenient to view the massive term as an interaction, the vertices m^2 have two attached lines and correspond to disks with two edges on the boundary. Thus the corresponding surfaces consist of spheres with one or two holes and two or four external edges. Let S be such surface. Note that S may have more than four external edges. We can say that an open subsurface $s \subset S$ is allowed if $\bar{s}$ is a disjoint union of 1PI surfaces and, in addition, each 1PI surface has two or four external edges. If we extract a 1PI surface with four (or two) external edges, then we insert a sphere with four (or two) external edges, this sphere has the same type as the other spheres with holes in the theory. By definition, the surface itself and the empty surface are allowed subsurfaces. We see that the set of surfaces corresponding to ribbon graphs in the theory is closed with respect to the operation of extraction of allowed subsurfaces. It is not difficult to check that the properties of the operation of extraction hold. Thus we can construct the Hopf algebra of such surfaces. This Hopf algebra will be denoted by $\mathcal{H}_4$.

The coproduct in $\mathcal{H}_4$ is

$$\Delta\Gamma = \Gamma \otimes \emptyset + \emptyset \otimes \Gamma + \sum_{\gamma\subset\Gamma}{}'_4 \bar{\gamma} \otimes \frac{\Gamma}{\gamma}\,, \tag{3.12}$$

the sum is over open $\gamma \subset \Gamma$ such that $\bar{\gamma}$ is a disjoint union of 1PI surfaces with two or four external edges.

4. Functions on surfaces. Renormalization

The linear functions on $\mathcal{H}$ form a linear space, denoted by $\mathcal{H}^*$.

The set of allowed surfaces $\{S_1, S_2 \ldots\}$ is formally a basis in $\mathcal{H}$. The basis in the dual space is given by linear functions S^i, $i \in \mathbb{N}$ such that

$$S^i(S_j) = \delta^i_j\,.$$

It is convenient to denote $S^i(S_j) = (S^i, S_j)$. We can say that the function S^i corresponds to the surface S_i, and that the space $\mathcal{H}^*$ is also a linear space of surfaces.

We use the coproduct in $\mathcal{H}$ to define a product in $\mathcal{H}^*$. The product is a mapping $\mu : \mathcal{H}^* \otimes \mathcal{H}^* \to \mathcal{H}^*$. The basis in $\mathcal{H}^* \otimes \mathcal{H}^*$ consists of the bilinear functions $S^i \otimes S^j$. The pairing of $S^i \otimes S^j$ with the basis vectors of $\mathcal{H} \otimes \mathcal{H}$ is the following

$$(S^i \otimes S^j,\, S_k \otimes S_l) = \delta^i_k \delta^j_l\,, \qquad \forall i,j,k,l \in \mathbb{N}.$$

The product $\mu : S^i \otimes S^j \mapsto S^i * S^j$ is defined by the formula

$$(S^i * S^j,\, S_k) = (S^i \otimes S^j,\, \Delta S_k)\,, \qquad \forall i,j,k \in \mathbb{N}\,,$$

or

$$S^i * S^j = \sum_k (S^i * S^j,\, S_k) S^k = \sum_k (S^i \otimes S^j,\, \Delta S_k) S^k\,. \tag{4.1}$$

We have $(S^i \otimes S^j,\, \Delta S_k) \neq 0$ if there is an open subsurface $s \subset S_k$ such that $\bar{s} = S_i$ and $S_k/s = S_j$. Thus we get S_j after the extraction of S_i from S_k.

In order to find the product $S^i * S^j$ we define the operation of insertion which is inverse to the operation of extraction. Let γ be a surface with m holes and R^i edges on the boundary of the i-th hole, $i = 1, \dots, m$. Also let Γ be a surface and let C be a sphere with holes inside Γ. If C has m holes and $\deg^i_\Gamma(C) = R^i$ $i = 1, \dots, m$, then we can insert $\mathrm{int}(\gamma)$ instead of $\mathrm{int}(C)$. We denote by $\gamma \circ \Gamma$ the new graph that contains γ instead of C. It is probably better to denote the graph $\gamma \circ \Gamma$ by $\gamma \circ^\alpha_c \Gamma$ in order to show that we insert γ instead of C. Sometimes, we can inset γ in different inequivalent ways. The index $'\alpha'$ denotes the way of insertion. But for shortness we use the notation $\gamma \circ \Gamma$.

Let $\mathring{\gamma} = \mathrm{int}(\gamma)$, we see that $(\gamma \circ \Gamma)/\mathring{\gamma} = \Gamma$.

If γ^* and Γ^* are dual to γ and Γ, then we have

$$(\gamma^* * \Gamma^*,\, \Gamma') = (\gamma^* \otimes \Gamma^*,\, \Delta\Gamma') = (\gamma^* \otimes \Gamma^*,\, \gamma \otimes \Gamma'/\mathring{\gamma} + \cdots) \neq 0\,.$$

In order to define the product more precisely we introduce a new formula for the coproduct. Let Γ be a surface. Sometimes we can find different open subsurfaces $\gamma,\, \gamma' \subset \Gamma$ such that $\bar{\gamma} \sim \bar{\gamma'}$ and $\Gamma/\gamma \sim \Gamma/\gamma'$. We denote by $N(\Gamma_1, \Gamma_2; \Gamma)$ the number of open subsurfaces $\gamma \subset \Gamma$ such that $\bar{\gamma} \sim \Gamma_1$ and $\Gamma/\gamma \sim \Gamma_2$. If there is no such subsurfaces, then $N(\Gamma_1, \Gamma_2; \Gamma) = 0$.

Example. Let a be a surface and let $\Gamma = a^n$, $\Gamma_1 = a^m$, $\Gamma_2 = a^{n-m}$, where $n \geq m$ and $n, m \in \mathbb{N}$. Then $N(a^m, a^{n-m}; a^n) = \binom{n}{m} = \frac{n!}{m!(n-m)!}$.

The new formula for the coproduct is

$$\Delta\Gamma = \sum_{\gamma \subset \Gamma}^{\nsim} N\left(\bar{\gamma}, \frac{\Gamma}{\gamma}; \Gamma\right) \bar{\gamma} \otimes \frac{\Gamma}{\gamma}\,,$$

the sign '$\nsim$' means that the sum is over open $\gamma \subset \Gamma$ that give different (inequivalent) $\bar{\gamma}$ and Γ/γ.

Now the product is

$$S^i * S^j = \sum_k (S^i \otimes S^j,\, \Delta S_k) S^k = \sum_k \sum_{\gamma \subset S^k}^{\nsim} N\left(\bar{\gamma}, \frac{S_k}{\gamma}; S_k\right) \left(S^i \otimes S^j,\, \bar{\gamma} \otimes \frac{S_k}{\gamma}\right) S^k\,,$$

and

$$S^i * S^j = \sum_k N(S_i,\, S_j;\, S_k) S^k\,.$$

JHEP05(2002)013

We see that the sum is over all surfaces that appear after the insertion of S^i into S^j, multiplied by some symmetry factors. We can introduce a new basis in order to get rid of the symmetry factors $N(S_i,\, S_j;\, S_k)$.

The new basis is $\widetilde{S^i} = Z(S_i)S^i$, where $Z(S_i)$ is the number if symmetries of the surface S_i (see appendix D).

Example. Let a be a surface without internal symmetries, then $Z(a^n) = n!$, $n = 1, 2, \ldots$

Let S^i and S^j be 1PI surfaces. In the new basis the product is

$$\widetilde{S^i} * \widetilde{S^j} = \sum_{s^i \circ s^j}^{\not\sim} (\widetilde{S^i \circ S^j}\,) + \widetilde{S^i} \cdot \widetilde{S^j}\,,$$

the sum is over all inequivalent insertions of S^i into S^j, the surface $S^i \cdot S^j$ is a disjoint union of S^i and S^j; the notation $(\widetilde{S^i \circ S^j}\,)$ means $Z(S_i \circ S_j)\; S^i \circ S^j$.

Since the coproduct 'Δ' is coassociative, the product '$*$' is associative. Thus we have the associative algebra $\mathcal{H}^*$ with respect to the product '$*$'.

Now we define the Lie algebra $\mathcal{L}$ with the basis labeled by the 1PI surfaces $\widetilde{S^i}$. The Lie bracket is

$$\begin{aligned}
[\widetilde{S^i}, \widetilde{S^j}]_* &= \widetilde{S^i} * \widetilde{S^j} - \widetilde{S^j} * \widetilde{S^i} \\
&= \sum_{s^i \circ s^j}^{\not\sim} (\widetilde{S^i \circ S^j}\,) + \widetilde{S^i} \cdot \widetilde{S^j} - \sum_{s^j \circ s^i}^{\not\sim} (\widetilde{S^j \circ S^i}\,) - \widetilde{S^j} \cdot \widetilde{S^i} \\
&= \sum_{s^i \circ s^j}^{\not\sim} (\widetilde{S^i \circ S^j}\,) - \sum_{s^j \circ s^i}^{\not\sim} (\widetilde{S^j \circ S^i}\,)\,.
\end{aligned}$$

We see that $[\widetilde{S^i}, \widetilde{S^j}]_* \in \mathcal{L}$ since it is a sum of 1PI surfaces $(\widetilde{S^i \circ S^j}\,)$ and $(\widetilde{S^j \circ S^i}\,)$. The Jacobi identity is satisfied, since the product '$*$' is associative. The universal enveloping algebra $U(\mathcal{L}) \simeq \mathcal{H}^*$. Thus the space of linear functions on $\mathcal{H}$ has the structure of the universal enveloping algebra of the Lie algebra $\mathcal{L}$.

Consider the Hopf algebra $\mathcal{H}_4$. The dual Lie algebra $\mathcal{L}^{(4)}$ spans the vectors corresponding to 1PI surfaces, constructed of spheres with two or four external edges. We represent $\mathcal{L}^{(4)}$ as a direct sum

$$\mathcal{L}^{(4)} = \mathcal{L}_1 \oplus \mathcal{L}_2\,, \tag{4.2}$$

where $\mathcal{L}_1$ spans the vectors corresponding to 1PI surfaces with two or four external edges.

Let $\Gamma_1,\, \Gamma_1' \in \mathcal{L}_1$ and $\Gamma_2,\, \Gamma_2' \in \mathcal{L}_2$. Since we cannot insert the surfaces $\Gamma_2,\, \Gamma_2'$ into any other surfaces, we have the following commutators

$$\begin{aligned}
[\Gamma_2,\, \Gamma_2']_* &= 0\,, \\
[\Gamma_1,\, \Gamma_2]_* &= \sum_{s^i \circ s^j}^{\not\sim} \Gamma_1 \circ \Gamma_2 \in \mathcal{L}_2\,, \\
[\Gamma_1,\, \Gamma_1']_* &= \sum_{s^i \circ s^j}^{\not\sim} \Gamma_1 \circ \Gamma_1' - \sum_{s^i \circ s^j}^{\not\sim} \Gamma_1' \circ \Gamma_1 \in \mathcal{L}_1\,,
\end{aligned}$$

where we assume that all the surfaces are in the new basis. We can write

$$
\begin{aligned}
[\mathcal{L}_2,\ \mathcal{L}_2] &= 0\,,\\
[\mathcal{L}_1,\ \mathcal{L}_2] &\subset \mathcal{L}_2\,,\\
[\mathcal{L}_1,\ \mathcal{L}_1] &\subset \mathcal{L}_1\,.
\end{aligned}
$$

We see that $\mathcal{L}_2$ resembles the Lie algebra of translations and $\mathcal{L}_1$ resembles the Lie algebra of rotations. The Poincaré group is a semi-direct product of translations by the action of rotations. If $G,\ G_1$ and G_2 are the Lie groups corresponding to the Lie algebras $\mathcal{L}^{(4)}, \mathcal{L}_1,\ \mathcal{L}_2$, then G is a semi-direct product of G_2 by the action of G_1 [1]. Since $[\mathcal{L}_2,\ \mathcal{L}_2] = 0$, the Lie group G_2 is abelian.

Now we proceed with the characters of $\mathcal{H}$. A character is a linear function $\chi:\ \mathcal{H} \to \mathcal{F}$ such that

$$
\chi(\Gamma_1 \cdot \Gamma_2) = \chi(\Gamma_1)\chi(\Gamma_2)\,, \qquad \forall\, \Gamma_1,\ \Gamma_2 \in \mathcal{H}\,.
$$

The set of characters has the structure of a group. We define the product of characters χ_1 and χ_2 by the formula

$$
\chi_1 * \chi_2(\Gamma) = \chi_1 \otimes \chi_2(\Delta\Gamma)\,, \qquad \forall\, \Gamma \in \mathcal{H}\,, \tag{4.3}
$$

where $\chi_1 \otimes \chi_2(\Gamma_1 \otimes \Gamma_2) = \chi_1(\Gamma_1)\chi_2(\Gamma_2)$ is a character of $\mathcal{H} \otimes \mathcal{H}$. In particular,

$$
\chi \otimes \chi(\Gamma_1 \otimes \Gamma_2) = \chi(\Gamma_1)\chi(\Gamma_2) = \chi(\Gamma_1 \cdot \Gamma_2)\,.
$$

The product $\chi_1 * \chi_2$ is a character, since $\Delta(\Gamma_1 \cdot \Gamma_2) = \Delta\Gamma_1 \cdot \Delta\Gamma_2$ and

$$
\begin{aligned}
\chi_1 * \chi_2(\Gamma_1 \cdot \Gamma_2) &= \chi_1 \otimes \chi_2(\Delta\Gamma_1 \cdot \Delta\Gamma_2)\\
&= \chi_1 \otimes \chi_2(\Delta\Gamma_1)\ \chi_1 \otimes \chi_2(\Delta\Gamma_2)\\
&= \chi_1 * \chi_2(\Gamma_1)\ \chi_1 * \chi_2(\Gamma_2)\,.
\end{aligned}
$$

The unit of the group is the counit of $\mathcal{H}$. We have

$$
\begin{aligned}
\varepsilon * \chi(\Gamma) &= \varepsilon \otimes \chi(\Delta\Gamma) = \varepsilon \otimes \chi(\emptyset \otimes \Gamma + \cdots)\\
&= \varepsilon(\emptyset)\ \chi(\Gamma) = \chi(\Gamma)\,,
\end{aligned}
$$

and similarly

$$
\chi * \varepsilon(\Gamma) = \chi(\Gamma)\,.
$$

The inverse character $\chi^{-1} = \chi \circ S$ or $\chi^{-1}(\Gamma) = \chi(S(\Gamma))$. We have

$$
\begin{aligned}
\chi^{-1} * \chi(\Gamma) &= \chi^{-1} * \chi(\Delta\Gamma) = \chi \otimes \chi((S \otimes id)\Delta\Gamma)\\
&= \chi(m(S \otimes id)\Delta\Gamma) = \delta_{\emptyset,\Gamma}\chi(\emptyset) = \delta_{\emptyset,\Gamma} = \varepsilon(\Gamma)\,.
\end{aligned}
$$

The axiom of the antipode gives

$$
\chi * \chi^{-1}(\Gamma) = \varepsilon(\Gamma)\,.
$$

Example. Exponentiation. Consider the Lie algebra $\mathcal{L}^{(4)}$ dual to the Hopf algebra $\mathcal{H}_4$. We have the decomposition $\mathcal{L}^{(4)} = \mathcal{L}_1 \oplus \mathcal{L}_2$ defined by (4.2). Let $a \in \mathcal{L}_2$ be a surface such that the symmetry factor $Z(a) = 1$. We define a^k to be a disjoint union of k surfaces a, by definition $a^0 = \emptyset$. Assume that a has no allowed subsurfaces except for the empty one and the surface itself. Then the coproduct of a^n is

$$\Delta a^n = \sum_{j=0}^{n} \binom{n}{j} a^j \otimes a^{n-j} \,.$$

The product $a^k * a^l$ in the dual algebra is

$$a^k * a^l = \left(a^k \otimes a^l, \Delta a^{k+l}\right) a^{k+l} = \binom{k+l}{l} a^{k+l} \,,$$

or

$$a^k * a^l = \frac{(k+l)!}{k!\, l!} a^{k+l} \,.$$

In the new basis $\widetilde{a^n} = n! a^n$. We have

$$\widetilde{a^k} * \widetilde{a^l} = \widetilde{a^{k+l}} \,.$$

We may complete the Hopf algebra by the formal series and define

$$\widetilde{e^a} = \sum_{n=0}^{\infty} \frac{1}{n!} \widetilde{a^n} \,.$$

Note that

$$\widetilde{e^a} = \sum_{n=0}^{\infty} \frac{1}{n!} \widetilde{a^n} = \sum_{n=0}^{\infty} a^n \,.$$

Now we prove that $\widetilde{e^a}$ is a character. Let $\Gamma_1,\ \Gamma_2 \in \mathcal{H}$, then

$$\left(\widetilde{e^a}, \Gamma_1 \cdot \Gamma_2\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\widetilde{a^n}, \Gamma_1 \cdot \Gamma_2\right) = \sum_{n=0}^{\infty} \delta_{a^n, \Gamma_1 \cdot \Gamma_2} \,.$$

If $\Gamma_1 = a^k$ and $\Gamma_2 = a^l$ for some k and l, then $(\widetilde{e^a}, \Gamma_1 \cdot \Gamma_2) = 1$, else $(\widetilde{e^a}, \Gamma_1 \cdot \Gamma_2) = 0$. Similarly, we have

$$\left(\widetilde{e^a}, \Gamma_1\right)\left(\widetilde{e^a}, \Gamma_2\right) = \sum_{i=0}^{\infty} \delta_{a^i, \Gamma_1} \sum_{j=0}^{\infty} \delta_{a^j, \Gamma_2} \,.$$

As before, we see that if $\Gamma_1 = a^k$ and $\Gamma_2 = a^l$ for some k and l, then $(\widetilde{e^a}, \Gamma_1)(\widetilde{e^a}, \Gamma_2) = 1$, else $(\widetilde{e^a}, \Gamma_1)(\widetilde{e^a}, \Gamma_2) = 0$. Consequently

$$\left(\widetilde{e^a}, \Gamma_1 \cdot \Gamma_2\right) = \left(\widetilde{e^a}, \Gamma_1\right)\left(\widetilde{e^a}, \Gamma_2\right) ,$$

end we see that $\widetilde{e^a}$ is a character.

Let $\mathcal{R}$ be a ring containing the field $\mathcal{F}$. Suppose that $\mathcal{R} = \mathcal{R}_- \oplus \mathcal{R}_+$ and

$$\begin{array}{ll} \mathcal{R}_- + \mathcal{R}_- \subset \mathcal{R}_- & \qquad \mathcal{R}_+ + \mathcal{R}_+ \subset \mathcal{R}_+ \\ \mathcal{R}_- \cdot \mathcal{R}_- \subset \mathcal{R}_- & \qquad \mathcal{R}_+ \cdot \mathcal{R}_+ \subset \mathcal{R}_+ \end{array}$$

Example. The ring of Laurent series without essential singularities in any finite region. The space $\mathcal{R}_-$ consists of polynomials in $1/z$ without constant terms

$$f_- = \sum_{i=1}^{n} a_i \left(\frac{1}{z}\right)^i , \qquad f_- \in \mathcal{R}_- .$$

The space $\mathcal{R}_+$ consists of series

$$f_+ = \sum_{i=0}^{\infty} a_i z^i , \qquad f_+ \in \mathcal{R}_+ .$$

We denote by $P_- : \mathcal{R} \to \mathcal{R}_-$ the projector on $\mathcal{R}_-$.

Let F be a character, $F : \mathcal{H} \to \mathcal{R}$. Note, that Feynman integrals in dimensional regularization are such characters. Indeed, if Γ_1 and Γ_2 are graphs-surfaces and $\Gamma_1 \cdot \Gamma_2$ is their disjoint union, then the Feynman integral $F(\Gamma_1 \cdot \Gamma_2) = F(\Gamma_1)F(\Gamma_2)$. Also in dimensional regularization Feynman integrals are Laurent series in the parameter of regularization.

We say that F_+ is a positive character if $F_+ : \mathcal{H} \to \mathcal{R}_+$. We say that F_- is a negative character if $(1 - P_-)F_-(\Gamma) = \delta_{\emptyset,\Gamma}$, i.e. $F_-(\emptyset) = 1$ and $P_-F_-(\Gamma) = F_-(\Gamma)$, $\forall \Gamma \neq \emptyset$.

Proposition 2. *For any character F there exists a positive character R such that $R = C * F$, where C is a negative character. Characters C and R depend on F. The product '$*$' is defined by (4.3).*

In terms of Feynman integrals, C is the counterterm, R is the renormalized value of the Feynman integral F.

We define C recursively.

$$C(\emptyset) = 1 .$$

Assume that we have found $C(\gamma)$ for all surfaces γ with the number of internal lines $P_\gamma \leq N$. Let Γ be a surface with $P_\Gamma = N + 1$. We have two conditions on C

$$\begin{cases} P_-(C * F)(\Gamma) = 0 ; \\ P_-C(\Gamma) = C(\Gamma) . \end{cases} \tag{4.4}$$

By the definition of the product

$$\begin{aligned} (C * F)(\Gamma) &= (C \otimes F)(\Delta\Gamma) \\ &= (C \otimes F) \left(\Gamma \otimes \emptyset + \emptyset \otimes \Gamma + \sum_{\gamma \subset \Gamma}{}' \bar{\gamma} \otimes \frac{\Gamma}{\gamma} \right) \\ &= C(\Gamma) \cdot F(\emptyset) + C(\emptyset) \cdot F(\Gamma) + \sum_{\gamma \subset \Gamma}{}' C(\bar{\gamma}) \cdot F\left(\frac{\Gamma}{\gamma}\right) . \end{aligned}$$

Applying conditions (4.4) we have

$$P_-(C * F)(\Gamma) = C(\Gamma) + P_- \left(F(\Gamma) + \sum_{\gamma \subset \Gamma}{}' C(\bar{\gamma}) \cdot F\left(\frac{\Gamma}{\gamma}\right) \right) = 0$$

and

$$C(\Gamma) = -P_{-}\left(F(\Gamma) + {\sum_{\gamma\subset\Gamma}}' C(\bar{\gamma})\cdot F\left(\frac{\Gamma}{\gamma}\right)\right). \tag{4.5}$$

Since in the sum on the right hand side $\gamma \subset \mathrm{int}(\Gamma)$ and $\gamma \neq \mathrm{int}(\Gamma)$, the number of internal edges $P_\gamma < P_\Gamma = N+1$, i.e. $P_\gamma \leq N$. Thus we have defined $C(\Gamma)$. Formula (4.5) is also called Bogolubov's recursion.

Proposition 3. *Let F be a character, then the function C defined by (4.5) is a character, i.e. $C(\Gamma_1\cdot\Gamma_2) = C(\Gamma_1)C(\Gamma_2)$, $\forall\Gamma_1, \Gamma_2$.*

The proof is given in appendix E.

Corollary 1. *The function R as a product of characters C and F is a character. Also we see that C is a negative character and R is a positive one.*

Example. Renormalizations of the theory Φ^4 with largangian (2.1).
Let Γ be a ribbon graph. The Feynman integral corresponding to Γ is divergent if

$$\omega_\Gamma = dL_\Gamma - 2P_\Gamma \geq 0\,,$$

where $d = 4$ is the dimension, $L_\Gamma = P_\Gamma - V_\Gamma + 1$ is the number of independent integrations, P_Γ is the numbers of internal lines, and V_Γ is the number of vertices. Formula (2.6) yields

$$4V_\Gamma = 2P_\Gamma + R_\Gamma\,,$$

where R_Γ is the number of external lines. Hence

$$\omega_\Gamma = 4(P_\Gamma+1) - 4V_\Gamma - 2P_\Gamma = 4 - R_\Gamma\,.$$

Thus the divergent integrals correspond to the graphs with $R_\Gamma \leq 4$. If $\Gamma \neq \emptyset$ and $R_\Gamma > 4$, then $C(\Gamma) = 0$. The renormalized Feynman integral

$$R(\Gamma) = (C\otimes F)(\Delta\Gamma) = \sum_{\gamma\subset\Gamma} C(\bar{\gamma})F\left(\frac{\Gamma}{\gamma}\right) = {\sum_{\gamma\subset\Gamma}}_4 C(\bar{\gamma})F\left(\frac{\Gamma}{\gamma}\right),$$

the last sum is over open $\gamma \subset \Gamma$ such that $\bar{\gamma}$ is a disjoint union of 1PI surfaces with two or four external edges, also we have $\bar{\gamma} = \emptyset$ and $\bar{\gamma} = \Gamma$ in the sum. We see that for the renormalization it suffices to construct the Hopf algebra $\mathcal{H}_4$.

Now we find the structure of the renormalization in the $1/N$ expansion. Graphs-surfaces which give the renormalization of the vertices g_1 contain four external edges and have the topology of disks with handles, hence by (2.8) we have

$$\Delta g_1^{(\mathrm{ren})} \sim N^{-\chi-2} = N^{-1-2g}\,,$$

where g is the number of handles. The surfaces corresponding to the vertices g_2 have the topology of cylinders with handles, consequently the renormalization

$$\Delta g_2^{(\mathrm{ren})} \sim N^{-\chi-2} = N^{-2-2g}\,.$$

Finally, we have

$$\begin{aligned} g_1^{(\mathrm{ren})} &= \frac{a_1}{N} + \frac{a_3}{N^3} + \frac{a_5}{N^5} + \cdots \,, \\ g_2^{(\mathrm{ren})} &= \frac{a_2}{N^2} + \frac{a_4}{N^4} + \frac{a_6}{N^6} + \cdots \,. \end{aligned}$$

A. Coassociativity

We have to prove

$$(id \otimes \Delta)\Delta\Gamma = (\Delta \otimes id)\Delta\Gamma \,, \qquad \forall \Gamma \,. \tag{A.1}$$

Using the left hand side:

$$\begin{aligned} (id \otimes \Delta)\Delta\Gamma &= (id \otimes \Delta) \sum_{\gamma \subset \Gamma} \bar{\gamma} \otimes \frac{\Gamma}{\gamma} \\ &= \sum_{\gamma_1 \subset \Gamma} \sum_{\gamma_2 \subset \frac{\Gamma}{\gamma_1}} \bar{\gamma}_1 \otimes \bar{\gamma}_2 \otimes \frac{\Gamma/\gamma_1}{\gamma_2} \,. \end{aligned}$$

We use the second property of the operation of extraction in order to find $\gamma_{12} \subset \Gamma$ such that $(\Gamma/\gamma_1)/\gamma_2 = \Gamma/\gamma_{12}$ and $\bar{\gamma}_{12}/\gamma_1 = \gamma_2$. Since the operation of extraction is bijective we can change the sum over $\gamma_2 \subset \Gamma/\gamma_1$ to the sum over $\gamma_{12} : \ \gamma_1 \subset \gamma_{12} \subset \Gamma$. After that we change the order of summation.

$$\begin{aligned} (id \otimes \Delta)\Delta\Gamma &= \sum_{\gamma_1 \subset \Gamma} \ \sum_{\gamma_{12} :\, \gamma_1 \subset \gamma_{12} \subset \Gamma} \bar{\gamma}_1 \otimes \frac{\bar{\gamma}_{12}}{\gamma_1} \otimes \frac{\Gamma}{\gamma_{12}} \\ &= \sum_{\gamma_{12} \subset \Gamma} \ \sum_{\gamma_1 \subset \bar{\gamma}_{12}} \bar{\gamma}_1 \otimes \frac{\bar{\gamma}_{12}}{\gamma_1} \otimes \frac{\Gamma}{\gamma_{12}} = (\Delta \otimes id) \sum_{\gamma_{12} \subset \Gamma} \bar{\gamma}_{12} \otimes \frac{\Gamma}{\gamma_{12}} = (\Delta \otimes id)\Delta\Gamma \,. \end{aligned}$$

Thus the axiom is proved.

B. The antipode as a homomorphism of the Hopf algebra

The antipode is defined by

$$m(S \otimes id)\Delta(\Gamma) = i\varepsilon(\Gamma) \,. \tag{B.1}$$

The coproduct of Γ is

$$\Delta\Gamma = \Gamma \otimes \emptyset + \sum_{\gamma \subset \Gamma}^{\neq} \bar{\gamma} \otimes \frac{\Gamma}{\gamma} \,,$$

where the sum is over all open $\gamma \subset \Gamma$ such that $\bar{\gamma} \neq \Gamma$; note that the number of internal edges $P_{\bar{\gamma}} < P_\Gamma$ in this case. Formula (B.1) gives the recursion

$$\begin{aligned} S(\emptyset) &= \emptyset \,, \\ S(\Gamma) &= -\sum_{\gamma \subset \Gamma}^{\neq} S(\bar{\gamma}) \cdot \frac{\Gamma}{\gamma} \Gamma \neq \emptyset \,. \end{aligned} \tag{B.2}$$

We must prove that $S(\Gamma_1 \cdot \Gamma_2) = S(\Gamma_1) \cdot S(\Gamma_2)$, $\forall \Gamma_1,\ \Gamma_2$.

We prove it by induction on the number of internal edges. If $P_{\Gamma_1 \cdot \Gamma_2} = 0$, then $\Gamma_1 = \Gamma_2 = \emptyset$ and $S(\Gamma_1 \cdot \Gamma_2) = S(\Gamma_1)S(\Gamma_2) = \emptyset$.

Assume that we have proved the assertion for all γ_1 and γ_2 such that $P_{\gamma_1 \cdot \gamma_2} \leq N$. Now we prove this for Γ_1 and Γ_2 such that $P_{\Gamma_1 \cdot \Gamma_2} = N + 1$. If $\Gamma_1 = \emptyset$ or $\Gamma_2 = \emptyset$ then the proof is trivial. Thus we assume that $\Gamma_1 \neq \emptyset$ and $\Gamma_2 \neq \emptyset$. Also since $\mathcal{H}$ is a bialgebra, $\Delta(\Gamma_1 \cdot \Gamma_2) = \Delta\Gamma_1 \cdot \Delta\Gamma_2$. By definition (B.1) we have

$$m(S \otimes id)(\Delta\Gamma_1 \cdot \Delta\Gamma_2) = 0\,,$$

thus

$$\begin{aligned}
0 &= m(S \otimes id)\left(\Gamma_1 \otimes \emptyset + \sum_{\gamma_1 \subset \Gamma_1}^{\neq} \bar{\gamma}_1 \otimes \frac{\Gamma_1}{\gamma_1}\right)\left(\Gamma_2 \otimes \emptyset + \sum_{\gamma_2 \subset \Gamma_2}^{\neq} \bar{\gamma}_2 \otimes \frac{\Gamma_2}{\gamma_2}\right) \\
&= S(\Gamma_1 \cdot \Gamma_2) + \sum_{\gamma_1 \subset \Gamma_1}^{\neq} S(\Gamma_2 \cdot \bar{\gamma}_1) \cdot \frac{\Gamma_1}{\gamma_1} + \sum_{\gamma_2 \subset \Gamma_2}^{\neq} S(\Gamma_1 \cdot \bar{\gamma}_2) \cdot \frac{\Gamma_2}{\gamma_2} \\
&\qquad + \sum_{\gamma_1 \subset \Gamma_1}^{\neq} \sum_{\gamma_2 \subset \Gamma_2}^{\neq} S(\bar{\gamma}_1 \cdot \bar{\gamma}_2) \cdot \frac{\Gamma_1}{\gamma_1} \cdot \frac{\Gamma_2}{\gamma_2} \\
&= S(\Gamma_1 \cdot \Gamma_2) + S(\Gamma_2) \cdot (-S(\Gamma_1)) + S(\Gamma_1) \cdot (-S(\Gamma_2)) + (-S(\Gamma_1)) \cdot (-S(\Gamma_2)) \\
&= S(\Gamma_1 \cdot \Gamma_2) - S(\Gamma_1) \cdot S(\Gamma_2)\,,
\end{aligned} \tag{B.3}$$

here we have used equations (B.2) and the inequalities $P_{\Gamma_1 \cdot \bar{\gamma}_2},\ P_{\Gamma_2 \cdot \bar{\gamma}_1},\ P_{\bar{\gamma}_1 \cdot \bar{\gamma}_2} < P_{\Gamma_1 \cdot \Gamma_2} = N + 1$. We see that $S(\Gamma_1 \cdot \Gamma_2) = S(\Gamma_1) \cdot S(\Gamma_2)$.

C. Axiom of the antipode

At first we discuss some general facts concerning surfaces and after that we prove the axiom.

Let Γ be a surface with decomposition. As always we denote by P_Γ the number of internal edges on Γ. Let $\{\gamma_0, \gamma_1, \ldots, \gamma_{n+1}\}$ be a set of open surfaces such that $\gamma_i \neq \gamma_j$ $\forall i \neq j$ and $\emptyset = \gamma_{n+1} \subset \gamma_n \subset \ldots \subset \gamma_0 = \mathrm{int}(\Gamma)$. Assume that for all $i = 0, \ldots, n$ there is no $\gamma \subset \Gamma$ such that $\gamma \neq \gamma_i,\ \gamma_{i+1}$ and $\gamma_i \subset \gamma \subset \gamma_{i+1}$, in this case the set $\{\gamma_0, \gamma_1, \ldots, \gamma_{n+1}\}$ will be called a maximal decreasing sequence of subsurfaces of Γ. Note that $n \leq P_\Gamma$, thus the sequence is finite.

Consider the mapping $\varphi_{\gamma_{i+1}} : \ \bar{\gamma}_i \to \bar{\gamma}_i/\gamma_{i+1}$. Since the mapping is bijective and the set $\{\gamma : \ \gamma_i \subset \gamma \subset \bar{\gamma}_{i+1}\}$ consists of the two elements $\{\bar{\gamma}_i,\ \bar{\gamma}_{i+1}\}$, we have $\{\widetilde{\gamma} : \ \widetilde{\gamma} \subset \bar{\gamma}_i/\gamma_{i+1}\} = \{\emptyset,\ \bar{\gamma}_i/\gamma_{i+1}\}$. Consequently

$$\Delta\left(\frac{\bar{\gamma}_i}{\gamma_{i+1}}\right) = \frac{\bar{\gamma}_i}{\gamma_{i+1}} \otimes \emptyset + \emptyset \otimes \frac{\bar{\gamma}_i}{\gamma_{i+1}}$$

and $S(\bar{\gamma}_i/\gamma_{i+1}) = -\bar{\gamma}_i/\gamma_{i+1}$, in particular $S(\bar{\gamma}_n) = -\bar{\gamma}_n$.

Also we shall use the following fact. If $\{\gamma_0, \gamma_1, \ldots, \gamma_{n+1}\}$ is a maximal decreasing sequence of subsurfaces of Γ, then

$$S(\bar{\gamma}_n) \cdot \frac{\bar{\gamma}_{n-1}}{\gamma_n} - \bar{\gamma}_n \cdot S\left(\frac{\bar{\gamma}_{n-1}}{\gamma_n}\right) = -\bar{\gamma}_n \cdot \frac{\bar{\gamma}_{n-1}}{\gamma_n} + \bar{\gamma}_n \cdot \frac{\bar{\gamma}_{n-1}}{\gamma_n} = 0\,. \tag{C.1}$$

We denote by M_Γ the number of elements in the longest decreasing sequence. Note that $M_\Gamma \leq P_\Gamma + 1$.

The axiom of the antipode is

$$m(S \otimes id)\Delta\Gamma = i\varepsilon(\Gamma) = m(id \otimes S)\Delta\Gamma\,.$$

The left equation is used for the definition of the antipode

$$\begin{aligned} S(\emptyset) &= \emptyset\,, \\ S(\Gamma) &= -\Gamma - {\sum}'_{\gamma\subset\Gamma} S(\bar{\gamma}) \cdot \frac{\Gamma}{\gamma}. \end{aligned} \tag{C.2}$$

We prove the right equation

$$m(id \otimes S)\Delta\Gamma = i\varepsilon(\Gamma)\,.$$

If $\Gamma = \emptyset$, then

$$m(id \otimes S)(\emptyset \otimes \emptyset) = \emptyset \cdot S(\emptyset) = \emptyset = i\varepsilon(\emptyset)\,.$$

Assume that $\Gamma \neq \emptyset$. We prove the assertion by induction. It is convenient to denote $\bar{\gamma}_0 = \Gamma$ and $\Sigma = m(id \otimes S)\Delta\bar{\gamma}_0$.

The first step of the induction is

$$\begin{aligned} \Sigma &= m(id \otimes S)\Delta\bar{\gamma}_0 = \bar{\gamma}_0 + S(\bar{\gamma}_0) + {\sum}'_{\gamma_1\subset\bar{\gamma}_0} S(\bar{\gamma}_1) \cdot \frac{\bar{\gamma}_0}{\gamma_1} \\ &= (-1){\sum}'_{\gamma_1\subset\bar{\gamma}_0} \left(S(\bar{\gamma}_1) \cdot \frac{\bar{\gamma}_0}{\gamma_1} - \bar{\gamma}_1 \cdot S\left(\frac{\bar{\gamma}_0}{\gamma_1}\right)\right), \end{aligned}$$

we have used recursive formula (C.2) for $S(\bar{\gamma}_0)$.

The second step is

$$\begin{aligned} \Sigma &= (-1){\sum}'_{\gamma_1\subset\bar{\gamma}_0}\left(\left\{-\bar{\gamma}_1 - {\sum}'_{\gamma_2\subset\bar{\gamma}_1} S(\bar{\gamma}_2) \cdot \frac{\bar{\gamma}_1}{\gamma_2}\right\} \cdot \frac{\bar{\gamma}_0}{\gamma_1} + \right. \\ &\qquad\qquad \left. + \bar{\gamma}_1 \cdot \left\{\frac{\bar{\gamma}_0}{\gamma_1} + {\sum}'_{\gamma'\subset\frac{\bar{\gamma}_0}{\gamma_1}} S(\bar{\gamma}') \cdot \frac{\bar{\gamma}_0/\gamma_1}{\gamma'}\right\}\right) \\ &= (-1)^2\left({\sum}'_{\gamma_1\subset\bar{\gamma}_0}{\sum}'_{\gamma_2\subset\bar{\gamma}_1} S(\bar{\gamma}_2) \cdot \frac{\bar{\gamma}_1}{\gamma_2} \cdot \frac{\bar{\gamma}_0}{\gamma_1} - \right. \\ &\qquad\qquad \left. - {\sum}'_{\gamma_1\subset\bar{\gamma}_0}{\sum}'_{\gamma'\subset\frac{\bar{\gamma}_0}{\gamma_1}} \bar{\gamma}_1 \cdot S(\bar{\gamma}') \cdot \frac{\bar{\gamma}_0/\gamma_1}{\gamma'}\right). \end{aligned}$$

Similarly, as in the proof of the coassociativity, we find γ_1' such that $(\bar{\gamma}_0/\gamma_1)/\gamma' = \bar{\gamma}_0/\gamma_1'$ and $\bar{\gamma}' = \bar{\gamma}_1'/\gamma_1$. After that we change the order of the summation in the second term

$${\sum}'_{\gamma_1\subset\bar{\gamma}_0}{\sum}'_{\gamma'\subset\frac{\bar{\gamma}_0}{\gamma_1}} \bar{\gamma}_1 \cdot S(\bar{\gamma}') \cdot \frac{\bar{\gamma}_0/\gamma_1}{\gamma'} = {\sum}'_{\gamma_1'\subset\bar{\gamma}_0}{\sum}'_{\gamma_1\subset\bar{\gamma}_1'} \bar{\gamma}_1 \cdot S\left(\frac{\bar{\gamma}_1'}{\gamma_1}\right) \cdot \frac{\bar{\gamma}_0}{\gamma_1'}\,.$$

If we change the notations $\gamma_1 \to \gamma_2$ and $\gamma'_1 \to \gamma_1$, then

$$\begin{aligned}\Sigma &= (-1)^2 {\sum}'_{\gamma_1\subset\bar\gamma_0} {\sum}'_{\gamma_2\subset\bar\gamma_1} \left(S(\bar\gamma_2)\cdot\frac{\bar\gamma_1}{\gamma_2}\cdot\frac{\bar\gamma_0}{\gamma_1} - \bar\gamma_2\cdot S\left(\frac{\bar\gamma_1}{\gamma_2}\right)\cdot\frac{\bar\gamma_0}{\gamma_1}\right)\\ &= (-1)^2 {\sum}'_{\gamma_1\subset\bar\gamma_0} {\sum}'_{\gamma_2\subset\bar\gamma_1} \left(S(\bar\gamma_2)\cdot\frac{\bar\gamma_1}{\gamma_2} - \bar\gamma_2\cdot S\left(\frac{\bar\gamma_1}{\gamma_2}\right)\right)\cdot\frac{\bar\gamma_0}{\gamma_1}.\end{aligned}$$

Assume that after the n-th step we have

$$\begin{aligned}\Sigma = (-1)^n {\sum}'_{\gamma_1\subset\bar\gamma_0}\cdots{\sum}'_{\gamma_n\subset\bar\gamma_{n-1}} \left(S(\bar\gamma_n)\cdot\frac{\bar\gamma_{n-1}}{\gamma_n} - \bar\gamma_n\cdot S\left(\frac{\bar\gamma_{n-1}}{\gamma_n}\right)\right)\times\\ \times\frac{\bar\gamma_{n-2}}{\gamma_{n-1}}\cdots\frac{\bar\gamma_0}{\gamma_1}.\end{aligned}$$

We repeat the reasonings of the second step and obtain

$$\begin{aligned}&{\sum}'_{\gamma_n\subset\bar\gamma_{n-1}} \left(S(\bar\gamma_n)\cdot\frac{\bar\gamma_{n-1}}{\gamma_n} - \bar\gamma_n\cdot S\left(\frac{\bar\gamma_{n-1}}{\gamma_n}\right)\right) =\\ &\quad= {\sum}'_{\gamma_n\subset\bar\gamma_{n-1}}{\sum}'_{\gamma_{n+1}\subset\bar\gamma_n} \left(S(\bar\gamma_{n+1})\cdot\frac{\bar\gamma_n}{\gamma_{n+1}} - \bar\gamma_{n+1}\cdot S\left(\frac{\bar\gamma_n}{\gamma_{n+1}}\right)\right)\cdot\frac{\bar\gamma_{n-1}}{\gamma_n}.\end{aligned}$$

Thus after the $(n+1)$-st step we have

$$\begin{aligned}\Sigma = (-1)^{n+1} {\sum}'_{\gamma_1\subset\bar\gamma_0}\cdots{\sum}'_{\gamma_{n+1}\subset\bar\gamma_n} \left(S(\bar\gamma_{n+1})\cdot\frac{\bar\gamma_n}{\gamma_{n+1}} - \bar\gamma_{n+1}\cdot S\left(\frac{\bar\gamma_n}{\gamma_{n+1}}\right)\right)\times\\ \times\frac{\bar\gamma_{n-1}}{\gamma_n}\cdots\frac{\bar\gamma_0}{\gamma_1}.\end{aligned}$$

Observe that the sums are over decreasing sequences of subsurfaces of Γ. If $n+2 = M_\Gamma$, then after the n-th step we have the sum only over maximal decreasing sequences $\{\gamma_0,\, \gamma_1, \dots,\, \gamma_{n+1}\}$, where $\gamma_{n+1} = \emptyset$ and $\gamma_0 = \mathrm{int}(\Gamma)$.
Since we have $S(\bar\gamma_n)\cdot\bar\gamma_{n-1}/\gamma_n - \bar\gamma_n\cdot S(\bar\gamma_{n-1}/\gamma_n) = 0$ in this case, we conclude that

$$\begin{aligned}\Sigma &= (-1)^n \sum_{\{\gamma_1,\dots,\gamma_n\}}{}' \left(S(\bar\gamma_n)\cdot\frac{\bar\gamma_{n-1}}{\gamma_n} - \bar\gamma_n\cdot S\left(\frac{\bar\gamma_{n-1}}{\gamma_n}\right)\right)\times\\ &\qquad\times\frac{\bar\gamma_{n-2}}{\gamma_{n-1}}\cdots\frac{\bar\gamma_0}{\gamma_1}\\ &= 0\,,\end{aligned}$$

thus the axiom is proved.

D. Symmetries of surfaces

Let Γ be a surface with decomposition, and let P_Γ be the number of internal edges. Let us enumerate the internal edges by the indices $i = 1, \dots, P_\Gamma$. We denote by Γ_I the surface Γ with the enumerated internal edges. The order of elements of the set I is important. Let $j:\,\Gamma_I \to \Gamma_{\sigma(I)}$ be a continuous automorphism of Γ. This automorphism does not move the boundary of Γ and maps internal edges I to $\sigma(I)$, where $\sigma(I)$ is a permutation of elements of I. We say that two automorphisms $j_1 : \Gamma_I \to \Gamma_{\sigma_1(I)}$ and $j_2 : \Gamma_I \to \Gamma_{\sigma_2(I)}$ are equivalent if $\sigma_1(I) = \sigma_2(I)$. We say that surfaces are equivalent $\Gamma'_I \sim \Gamma''_J$ if there is an automorphism $j : \Gamma'_I \to \Gamma''_J$, i.e. $\Gamma' = \Gamma''$ and $J = \sigma(I)$. For brevity, sometimes we shall omit the subscript $\sigma(I)$. We denote by $\mathrm{Z}(\Gamma)$ the number of continuous automorphisms $j : \Gamma_I \to \Gamma_{\sigma(I)}$ up to the equivalence. We also say that j is a symmetry of Γ and that $\mathrm{Z}(\Gamma)$ is the number of the symmetries of Γ.

Consider the following examples of surfaces without boundary.

Example 1. Let Γ be a sphere with two internal edges. We can permute these edges, thus there are two automorphisms $j_1 : \Gamma_{\{1,2\}} \to \Gamma_{\{1,2\}}$ and $j_2 : \Gamma_{\{1,2\}} \to \Gamma_{\{2,1\}}$. In this case $\mathrm{Z}(\Gamma) = 2$.

Example 2. Let Γ be a sphere with a 'square'. Then we can rotate this 'square', thus there are four automorphisms

$$\begin{aligned} &j_1 : \Gamma_{\{1,2,3,4\}} \to \Gamma_{\{1,2,3,4\}}\,, && j_2 : \Gamma_{\{1,2,3,4\}} \to \Gamma_{\{2,3,4,1\}}\,, \\ &j_3 : \Gamma_{\{1,2,3,4\}} \to \Gamma_{\{3,4,1,2\}} \qquad \text{and} && j_4 : \Gamma_{\{1,2,3,4\}} \to \Gamma_{\{4,1,2,3\}}\,. \end{aligned}$$

The number of symmetries $\mathrm{Z}(\Gamma) = 4$.

Let Γ_1 and Γ_2 be surfaces such that $N(\Gamma_1, \Gamma_2; \Gamma) \neq 0$. As above $N(\Gamma_1, \Gamma_2; \Gamma)$ is the number of open subsurfaces $\gamma \subset \Gamma$ such that $\bar{\gamma} \sim \Gamma_1$ and $\Gamma/\gamma \sim \Gamma_2$. By definition, we have $n = N(\Gamma_1, \Gamma_2; \Gamma)$ open subsurfaces $\gamma_i \subset \Gamma$ such that $\bar{\gamma}_i \sim \Gamma_1$ and $\Gamma/\gamma_i \sim \Gamma_2$, $i = 1, \ldots, n$. Hence we have automorphisms $j_i : \Gamma/\gamma_1 \to \Gamma/\gamma_i$. One can find the corresponding automorphisms $j'_i : \Gamma \to \Gamma$. The automorphism j'_i takes the subsurface γ_1 to the subsurface γ_i. All other automorphisms $j : \Gamma \to \Gamma$ do not change the boundary of γ_1. They are either the internal automorphisms of γ_1 or the automorphisms of Γ that do not change γ_1. The last ones do not depend on γ_1 and remain in Γ/γ_1. Sometimes new symmetries appear after the extraction of a subsurface. Let γ, Γ and Γ' be surfaces such that $\gamma \circ \Gamma = \Gamma'$. Suppose that there are two insertions $\gamma_K \circ \Gamma_J = \Gamma'_I$ and $\gamma_K \tilde{\circ} \Gamma_{\sigma(J)} = \Gamma'_{\sigma'(I)}$, where $I = \{1, \ldots, P_{\Gamma'}\}$, $J = \{1, \ldots, P_\Gamma\}$ and $K = \{P_\Gamma + 1, \ldots, P_{\Gamma'}\}$. Assume that there exists $j : \Gamma_J \to \Gamma_{\sigma(J)}$ and there is no $j' : \Gamma'_I \to \Gamma'_{\sigma'(I)}$. Thus we have a new symmetry in Γ after the extraction of γ from Γ'. We denote by $N_C(\gamma, \Gamma; \Gamma')$ the number of such symmetries. This number is equal to the number of all inequivalent insertions $\gamma \circ \Gamma$. If $\Gamma' = \gamma \cdot \Gamma$, then we define $N_C(\gamma, \Gamma; \Gamma') = 1$.

Consequently the number of symmetries

$$Z(\Gamma) = Z(\bar{\gamma}_1) \frac{Z(\Gamma/\gamma_1)}{N_C(\bar{\gamma}_1, \Gamma/\gamma_1; \Gamma)} N\left(\bar{\gamma}_1, \frac{\Gamma}{\gamma_1}; \Gamma\right)$$

or

$$Z(\Gamma) N_C(\Gamma_1, \Gamma_2; \Gamma) = Z(\Gamma_1) Z(\Gamma_2)\; N(\Gamma_1, \Gamma_2; \Gamma)\,.$$

The new basis is $\widetilde{S^i} = Z(S_i) S^i$. Let S^i and S^j be 1PI surfaces, then the product

$$\begin{aligned} Z(S_i) S^i * Z(S_j) S^j &= \sum_k Z(S_i) Z(S_j)\; N(S_i, S_j; S_k) S^k \\ &= \sum_k Z(S_k)\; N_C(S_i, S_j; S_k) S^k \\ &= Z(S_i) Z(S_j)\; S^i \cdot S^j + \sum_{s^i \circ s^j}^{\not\sim} Z(S_i \circ S_j)\; S^i \circ S^j\,, \end{aligned}$$

JHEP05(2002)013

the last sum is over all inequivalent insertions $S^i \circ S^j$, the surface $S^i \cdot S^j$ is a disjoint union of S^i and S^j. We have

$$\widetilde{S^i} * \widetilde{S^j} = \sum_{s^i \circ s^j}^{\not\sim} (\widetilde{S^i \circ S^j}) + \widetilde{S^i} \cdot \widetilde{S^j} \,,$$

where the notation $(\widetilde{S^i \circ S^j})$ means $Z(S_i \circ S_j)\; S^i \circ S^j$.

E. Negative character

We have to prove that the function C defined by (4.5) is a character, i.e. $C(\Gamma_1 \cdot \Gamma_2) = C(\Gamma_1)C(\Gamma_2)$, $\forall \Gamma_1, \Gamma_2$.

As in appendix B, we prove the proposition by induction on the number of internal lines. If $P_{\Gamma_1 \cdot \Gamma_2} = 0$, then $\Gamma_1 = \Gamma_2 = \emptyset$ and $C(\Gamma_1 \cdot \Gamma_2) = C(\Gamma_1)C(\Gamma_2) = 1$.

Assume that we have proved the assertion for all γ_1 and γ_2 such that $P_{\gamma_1 \cdot \gamma_2} \leq N$. We have to prove it for Γ_1 and Γ_2 such that $P_{\Gamma_1 \cdot \Gamma_2} = N + 1$. If $\Gamma_1 = \emptyset$ or $\Gamma_2 = \emptyset$ then the proof is trivial. Thus we assume that $\Gamma_1 \neq \emptyset$ and $\Gamma_2 \neq \emptyset$.

By definition (4.5)

$$C(\Gamma) = -P_{-}\left(F(\Gamma) + {\sum_{\gamma \subset \Gamma}}' C(\bar{\gamma}) \cdot F\left(\frac{\Gamma}{\gamma}\right)\right) = -P_{-}\left(\sum_{\gamma \subset \Gamma}^{\neq} C(\bar{\gamma}) \cdot F\left(\frac{\Gamma}{\gamma}\right)\right),$$

the last sum is over $\gamma \subset \Gamma$, $\bar{\gamma} \neq \Gamma$. Consequently

$$\sum_{\gamma \subset \Gamma}^{\neq} C(\bar{\gamma}) \cdot F\left(\frac{\Gamma}{\gamma}\right) = -C(\Gamma) + C_{+}(\Gamma)\,,$$

where $C_{+}(\Gamma)$ is a positive function.

Now

$$\begin{aligned}
0 &= P_{-}(C * F)(\Gamma_1 \cdot \Gamma_2) \\
&= P_{-}\left\{(C \otimes F)(\Delta\Gamma_1 \cdot \Delta\Gamma_2)\right\} \\
&= P_{-}\left\{(C \otimes F)\left(\Gamma_1 \otimes \emptyset + \sum_{\gamma_1 \subset \Gamma_1}^{\neq} \bar{\gamma}_1 \otimes \frac{\Gamma_1}{\gamma_1}\right) \times \right. \\
&\qquad\qquad \left. \times \left(\Gamma_2 \otimes \emptyset + \sum_{\gamma_2 \subset \Gamma_2}^{\neq} \bar{\gamma}_2 \otimes \frac{\Gamma_2}{\gamma_2}\right)\right\} \\
&= P_{-}\left\{C(\Gamma_1 \cdot \Gamma_2) + \right. \\
&\qquad + \sum_{\gamma_2 \subset \Gamma_2}^{\neq} C(\Gamma_1 \cdot \bar{\gamma}_2) F\left(\frac{\Gamma_2}{\gamma_2}\right) + \\
&\qquad + \sum_{\gamma_1 \subset \Gamma_1}^{\neq} C(\Gamma_2 \cdot \bar{\gamma}_1) F\left(\frac{\Gamma_1}{\gamma_1}\right) +
\end{aligned}$$

$$+\sum_{\gamma_1\subset\Gamma_1}^{\neq}\sum_{\gamma_2\subset\Gamma_2}^{\neq} C(\bar{\gamma_1}\cdot\bar{\gamma_2})F\left(\frac{\Gamma_1}{\gamma_1}\cdot\frac{\Gamma_2}{\gamma_2}\right)\Bigg\}$$

$$\begin{aligned}
&= P_-\Big\{C(\Gamma_1\cdot\Gamma_2)+\\
&\qquad + C(\Gamma_1)\Big(-C(\Gamma_2)+C_+(\Gamma_2)\Big)+\\
&\qquad + C(\Gamma_2)\Big(-C(\Gamma_1)+C_+(\Gamma_1)\Big)+\\
&\qquad + \Big(-C(\Gamma_1)+C_+(\Gamma_1)\Big)\Big(-C(\Gamma_2)+C_+(\Gamma_2)\Big)\Big\}\\
&= P_-\left\{C(\Gamma_1\cdot\Gamma_2)-C(\Gamma_1)C(\Gamma_2)+C_+(\Gamma_1)C_+(\Gamma_2)\right\}\\
&= C(\Gamma_1\cdot\Gamma_2)-C(\Gamma_1)C(\Gamma_2)\,.
\end{aligned}$$

Consequently

$$C(\Gamma_1\cdot\Gamma_2)=C(\Gamma_1)C(\Gamma_2)\,,$$

and the assertion is proved.

Acknowledgments

The author is grateful to Alexey Morozov and Konstantin Selivanov for enormous help on all stages of this project and to S. Klevtsov, A. Dymarsky, K. Trushkin and R. Anno for useful discussions. Special thanks to A. Mironov and S. Rychkov for careful reading of the manuscript. The work is partially supported by RFBR grant 00-02-16477 and Russian president's grant 00-15-99296.

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